Appendix B

COMPLEX VARIABLES, THE SPECTRUM, AND LAG OPERATOR

In this Appendix, we review some basic results of complex variables, and their applications to the lag operator and spectral analysis. Section B.1 collects standard results of complex variables without proofs. Since most results in textbooks of complex variables are not relevant for our purpose, it is useful to collect the results used in macroeconometrics. Section B.2 gives examples of Hilbert spaces on C. We will use a Hilbert space in this section in order to define the spectrum and give a foundation for using lag operator methods. Section B.3 uses these results to prove important results involving the lag operator such as convergence conditions for infinite order AR and MA representations and invertibility conditions of AR and MA representations. The results relating to a removable singular point in Section B.1 are used to derive the Beveridge-Nelson decomposition (Section 13.2) and the nonlinear restrictions by the linear rational expectations models presented in Chapter 16. Section B.3 presents some results for the spectrum, using the tools developed in Sections B.1 and B.2.[1]

[1]Some of the results in Sections B.2 and B.3 can be found in Sargent (1987). The main difference between Sargent's presentation and the presentation here lies in the difference in the convergence

B.1 Complex Variables

B.1.1 Complex Numbers

A complex number $z = x + iy$ can be defined as ordered pairs (x, y) of real numbers, where i is a pure imaginary number that satisfies $i^2 = -1$. The real numbers x and y are known as the *real* and *imaginary parts* of z, respectively. It is natural to associate the complex number with a point in the plane whose Cartesian coordinates are x and y. In other words, each complex number corresponds to just one point. When used for the purpose of displaying the numbers $z = x + iy$ geometrically, the xy plane is called the *complex plane* C. We denote the complex number which corresponds to the origin of the complex plane by 0.

The *absolute value*, or *modulus*, of a complex number $z = x + iy$ is defined as $\sqrt{x^2 + y^2}$ and is denoted by $|z|$. The *complex conjugate* of a complex number $z = x+iy$ is defined as the complex number $x - iy$ and is denoted by $\overline{z}$. An important identity relating the conjugate of a complex number z to its absolute value is $z\overline{z} = |z|^2$.

A *circle with center at* z_0 and radius ϵ is $\{z : z$ is complex number such that $|z - z_0| = \epsilon\}$. The interior points of the circle are called the ϵ *neighborhood* of z_0. For any real number θ, it is convenient to define $e^{i\theta}$, or $\exp(i\theta)$, by

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{B.1}$$

Then $\overline{e^{i\theta}} = \cos\theta - i\sin\theta = e^{-i\theta}$, and $|e^{i\theta}| = \sqrt{e^{i\theta}e^{-i\theta}} = 1$. Hence $e^{i\theta}$ represents the circle with the center at the origin and radius of one. This circle is called the *unit*

concept for the z transform. Our definition allows us to use the results in Section B.1 for the z transform, which can be used to prove various results such as the condition for invertibility of a lag polynomial in terms of the zeros of the z transform.

circle. We can express any complex number in *exponential form*:

$$z = re^{i\theta}. \tag{B.2}$$

B.1.2 Analytic Functions

For a sequence of complex numbers $\{z_i\}_{i=1}^{\infty}$ and an infinite series of complex numbers $\sum_{i=1}^{\infty} z_i$, convergence and divergence are defined in the same way as those of real numbers except that the distance for complex numbers is used for the definitions. The series $\sum_{i=1}^{\infty} z_i$ is *absolutely convergent* if the series $\sum_{i=1}^{\infty} |z_i|$ of real numbers converges. Absolute convergence of a series of complex numbers implies the convergence of that series.

A complex-valued *function* f, defined on a set of complex numbers D, assigns a complex number $f(z)$ to each z in D. The set D is the *domain of definition* of f. A specific value of z for which $f(z) = 0$ is called a *zero* of a function f.

If n is a nonnegative integer, and if $a_0, a_1, a_2, \cdots, a_n$ are complex constants, where $a_n \neq 0$, the function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is a *polynomial* of degree n. Any polynomial of degree n has precisely n zeros as in the following proposition:

Proposition B.1.1 *(The Fundamental Theorem of Algebra)* For any polynomial of degree n, $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ where $n \geq 1$, there exist n complex numbers $z_1, z_2, \cdots, z_n$, such that

$$P(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n).$$

∎

Here z_i is a zero of $P(z)$, and a root of $P(z) = 0$. Note that z_i may be equal to z_j for some j.

The limits, continuity, derivatives, and differentiability of functions are defined in the same way as those of real-valued functions of a real variable except that the distance for complex numbers is used for the definitions. For example, for a function f with domain S

$$\lim_{z \to z_0} f(z) = w_0 \tag{B.3}$$

means that for each positive number ϵ there is a positive number δ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$ and $z \in S$. Similarly, the *derivative* of f at z_0 is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{B.4}$$

provided this limit exists. The function f is said to be *differentiable* at z_0 if its derivative at z_0 exists. Since it is possible to approach z_0 from many directions on the complex plane, the differentiability of functions of complex numbers is in a sense stricter than the differentiability of functions of real numbers as in the next example:

Example B.1.1 Let $f(z) = |z|^2$. Churchill and Brown (1984, p.40) show that $f(z)$ is differentiable only at the origin. For $z = x + iy$, let the real and imaginary parts of $f(z)$ be $u(x, y)$ and $v(x, y)$: $f(z) = u(x, y) + iv(x, y)$. Then $u(x, y) = x^2 + y^2$, and $v(x, y) = 0$. Hence even when the real and imaginary components of a function of a complex variable have continuous derivatives at z_0, the function may not be differentiable there. ∎

Since the definition of a derivative in (B.4) is identical to that of the derivative of a real-valued function of a real variable, most of the basic differentiation formulas remain valid for functions of complex variables. For example, if n is a positive integer, $\frac{dz^n}{dz} = nz^{n-1}$. This formula remains valid when n is a negative integer as long as $z \neq 0$.

A function f of the complex number z is *analytic* at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighborhood of z_0. An *entire function* is a function that is analytic at each point in the entire complex plane. Every polynomial is an *entire function*.

If two functions $f(z)$ and $g(z)$ are analytic in a domain D, then their sum and their product are both analytic in D. The quotient $\frac{f(z)}{g(z)}$ is also analytic in D provided that $g(z) \neq 0$ for any z in D. Hence the quotient $\frac{P(z)}{Q(z)}$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

The following three propositions are important for our purposes. See Churchill and Brown (1984, p.113, p.126, and p.153) for proofs.

Proposition B.1.2 Let a function f be analytic at a point z_0 of f. There is a neighborhood of z_0 throughout which f has no other zeros, unless f is identically zero. That is, the zeros of an analytic function which is not identically zero are isolated. ∎

Proposition B.1.3 If a function f is analytic at a point, then its derivatives of all orders exist and are themselves analytic there. ∎

Proposition B.1.4 *(Taylor's Theorem)* Let f be analytic everywhere inside a circle C with center at z and radius R. Then at each point z inside C,

$$\text{(B.5)} \qquad \begin{aligned} f(z) &= f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \\ &+ \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots . \end{aligned}$$

∎

The special case of series (B.5) when $z_0 = 0$ is called the *Maclaurin series.*

Example B.1.2 This example provides a Maclaurin series representation. Let $f(z) = \frac{1}{1-az}$ for a nonzero real number a. Then $f(z)$ is analytic on the complex plane except for $z = a^{-1}$.

(B.6)
$$f^{(n)}(z) = \frac{n!a^n}{(1-az)^{n+1}}$$

At each point z such that $|z| < |a^{-1}|$,

$$\frac{1}{1-az} = 1 + az + a^2z^2 + \cdots + a^nz^n + \cdots .$$

■

Let $S(z)$ be a power series:

$$S(z) = \sum_{n=0}^{\infty} a_n z^n.$$

See Churchill and Brown (1984, p.137 and p.143) for proofs of the following two propositions.

Proposition B.1.5 If the power series converges when $z = z_1$ ($z_1 \neq 0$), it is absolutely convergent for every value of z such that $|z| < |z_1|$. ■

The greatest circle about the origin such that the series converges at each point inside is called the *circle of convergence* of the power series.

Proposition B.1.6 The power series $S(z)$ represents a function that is analytic at every point in the interior of its circle of convergence. ■

If $S(z)$ converges for z such that $|z| < R$, then $S(z - z_0)$ is analytic for z such that $|z - z_0| < R$ because it is a composite function of two analytic functions.

When $f(z)$ is analytic for all z such that $|z - z_0| < R$ but fails to be analytic at z_0, then we cannot apply Taylor's theorem at that point. However, we can find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$. If $f(z)$ is analytic in the domain of all points z such that $R_1 \leq |z - z_0| \leq R_2$, then

$$f(z) = \sum_{n=0}^{\infty} a_0 (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^{-n} \tag{B.7}$$

in the domain. The series here is called a *Laurent series*. A series representation of this type is unique (see Churchill and Brown, 1984, pp. 132-134 and p.148).

When all the coefficients b_n in (B.7) are zero, the point z_0 is called a *removable singular point* of f. In this case, the Laurent series (B.7) contains only nonnegative powers of $z - z_0$. If we define $f(z)$ as a_0 at z_0, the function becomes analytic at z_0.

Suppose that a function can be written in the form

$$f(z) = \frac{g(z)}{z - z_0}, \tag{B.8}$$

where $g(z)$ is analytic everywhere inside a circle C with center at z_0 and radius R. Then at each point z inside C, $f(z)$ is analytic for all z except for $z = z_0$. From the Taylor series

$$\begin{aligned} g(z) &= g(z_0) + \frac{g'(z_0)}{1!}(z - z_0) + \frac{g''(z_0)}{2!}(z - z_0)^2 + \cdots \\ &+ \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots . \end{aligned} \tag{B.9}$$

It follows that

$$\begin{aligned} f(z) &= \frac{g(z_0)}{z - z_0} + \frac{g'(z_0)}{1!} + \frac{g''(z_0)}{2!}(z - z_0) + \cdots \\ &+ \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n-1} + \cdots . \end{aligned} \tag{B.10}$$

Then a is a nonzero real number, and $S(z)$ is a polynomial or a power series which converges for all z such that $|z| < R$ for some R. Hence if $g(z_0) = 0$, then z_0 is a removable singular point of $f(z)$.

B.2 Hilbert Spaces on C

In Appendix 3.A, it was noted that the complex plane, C, can be used as the set of scalars K for a vector space, and therefore for a Hilbert space. This section gives examples of Hilbert spaces for which $K = C$. The space of complex-valued random variables explained in Example B.2.4 and $L^2(Prob)$ of real-valued random variables explained in Appendix 3.A are the two Hilbert spaces we use in this book.

Example B.2.1 The complex plane, C, is a vector space on $K = C$ with addition and scalar multiplication defined in the usual way. When the norm of a complex number is defined as its absolute value, C is a Banach space. When the inner product is defined as $(x|y) = x\bar{y}$, C is a Hilbert space. ∎

Example B.2.2 Vectors in the space consist of sequences of n complex numbers, C^n, is a vector space on C when $\mathbf{x}+\mathbf{y}$ for $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)'$ is defined by $(x_1+y_1, x_2+y_2, \cdots, x_n+y_n)'$ and $\alpha\mathbf{x}$ for α in C is defined by $(\alpha x_1, \alpha x_2, \cdots, \alpha x_n)'$. When we define a norm of $\mathbf{x}$ as $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$, C^n is a Banach space. When we define $(\mathbf{x}|\mathbf{y}) = \sum_{i=1}^{n} x_i\bar{y}_i$, C^n is a Hilbert space on C. ∎

Example B.2.3 The space l_2 consists of all sequences of complex numbers $\{x_1, x_2, \cdots\}$ for which $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. The inner product of elements $\mathbf{x} = \{x_1, x_2, \cdots\}$ and $\mathbf{y} = \{y_1, y_2, \cdots\}$ in l_2 is defined as $(\mathbf{x}|\mathbf{y}) = \sum_{i=1}^{\infty} x_i\bar{y}_i$. With this inner product, l_2 is a Hilbert space on C. ∎

Example B.2.4 On the interval $[-\pi, \pi]$, use the uniform distribution to define the probability of the σ-field of the Borel sets in the interval. On this probability space, consider a complex-valued random variable $z = x + iy$ where x and y are real-valued random variables on $[-\pi, \pi]$. Define

$$E(z) = E(x) + iE(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(\lambda)d\lambda + i\frac{1}{2\pi}\int_{-\pi}^{\pi} x(\lambda)d\lambda$$

Let $L^2[-\pi, \pi] = \{h$: h is a complex valued random variable on $[-\pi, \pi]$ and $E(|h|^2) < \infty\}$. Then with an inner product defined by $(h_1|h_2) = E(h_1\bar{h}_2)$, $L^2[-\pi, \pi]$ is a Hilbert space. As in $L^2(Prob)$, if two different random variables h_1 and h_2 satisfy $E[|h_1 - h_2|^2] = 0$, then we view h_1 and h_2 as the same element in this space.[2] ∎

B.3 Spectrum

This section defines the spectral density for a linearly regular covariance stationary process. We will first consider stochastic processes of random variables. Then we will consider stochastic processes of random vectors.

Imagine that we are given a white noise process $\{v\}_{t=-\infty}^{\infty}$ on a probability space $(S, \mathcal{F}, Prob)$ that satisfies $E(v_t^2) = \sigma_v^2$ and $E(v_t v_s) = 0$ for $t \neq s$. It is convenient to normalize this white noise process by defining $e_t = \frac{v_t}{\sigma_v}$. Then $\{e_t\}_{t=-\infty}^{\infty}$ is an orthonormal sequence in $L^2(Prob)$ because it satisfies $\|e_t\| = \sqrt{E(e_t^2)} = 1$ and $(e_t|e_s) = E(e_t e_s) = 0$ for $t \neq s$. Let $b(L) = b_0 + b_1 L + b_2 L^2 + \cdots$ be a series in the lag operator. Then from Proposition 3.A.5, $b(L)e_t$ converges to an element in $L^2(Prob)$ if and only if $\{b_j\}_{j=1}^{\infty}$ is square summable, that is, $\sum_{j=1}^{\infty} |b_j|^2 < \infty$.

[2] For our purpose, it is convenient to view an element of $L^2[-\pi, \pi]$ as a complex-valued random variable when the uniform distribution is given on $[-\pi, \pi]$. In many books, this interpretation is not given, and elements in $L^2[-\pi, \pi]$ are considered complex-valued functions, f, which are measurable on $[-\pi, \pi]$.

Given an orthonormal sequence $\{e_t\}_{t=-\infty}^{\infty}$ in $L^2(Prob)$, imagine that we are interested in certain properties of $b(L)e_t$ for various series in the lag operator $b(L) = b_0 + b_1 L + b_2 L^2 + \cdots$ such as convergence of $b(L)e_t$ and the autocovariance of $b(L)e_t$. As long as these properties do not depend on the probability space, we can choose a probability space that makes studying these properties convenient. As we will see, it is convenient to consider a random variable and an orthonormal sequence in $L^2[-\pi, \pi]$ in which the probability is given by the uniform distribution on $[-\pi, \pi]$.

For this purpose, we consider a sequence $\{u_t\}_{t=-\infty}^{\infty}$ in $L^2[-\pi, \pi]$ where $u_t(\lambda) = \exp(-i\lambda t) = \cos(\lambda t) - i\sin(\lambda t)$. Then $|u_t(\lambda)| = 1$ for all λ in $[-\pi, \pi]$, so that $\|u\| = \sqrt{E(|u_t|^2)} = 1$. If $t \neq s$,

$$
\text{(B.11)} \qquad
\begin{aligned}
(u_t|u_s) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-i\lambda t)\overline{\exp(-i\lambda s)}d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-i\lambda t)\exp(i\lambda s)d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(i\lambda(s-t))d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(\lambda(s-t)) + i\sin(\lambda(s-t))]d\lambda \\
&= 0.
\end{aligned}
$$

Thus $\{u_t\}_{t=-\infty}^{\infty}$ is an orthonormal sequence.

Given $b(L)e_t = \sum_{j=0}^{\infty} b_j e_{t-j}$, consider a process $b(L)u_t = \sum_{j=0}^{\infty} b_j u_{t-j}$ in $L^2[-\pi, \pi]$. From Proposition 3.A.5, $b(L)e_t$ and $b(L)u_t$ converge if and only if $\{b_j\}$ is square summable. Hence $b(L)e_t$ converges if and only if $b(L)u_t$ converges.

Let M be the closed subspace in $L^2[-\pi, \pi]$ generated by $\{u_t\}_{t=-\infty}^{\infty}$. From Proposition 3.A.6, for any element y in M,

$$y = \sum_{j=0}^{\infty} c_j \exp(i\lambda j) \tag{B.12}$$

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(\lambda)\exp(-i\lambda j)d\lambda \tag{B.13}$$

where c_j is the Fourier coefficient for $u(-j) = \exp(i\lambda j)$ and $\{c_j\}$ is square summable.

When $\{b_j\}_{j=0}^{\infty}$ is square summable, let $x_t = b(L)e_t$. Then the autocovariance $\Phi(k) = E(x_t x_{t-k})$ is given by $\Phi(k) = \lim_{n\to\infty} E\left[(\sum_{j=0}^{n} b_j e_{t-j})(\sum_{j=0}^{n} b_j e_{t-k-j})\right] = \sum_{j=k}^{\infty} b_j b_{j-k}$, where the last equality can be proved by the continuity of the inner product (Proposition 3.A.2).

Define the autocovariance of order k, $\Phi(k)$ for $h_t = \sum_{j=0}^{\infty} b_j \exp(-i\lambda(t-j)) = \sum_{j=0}^{\infty} b_j \exp(i\lambda j)\exp(-i\lambda t) = h_0 \exp(-i\lambda t)$ as

$$\Phi(k) = E(h_t \bar{h}_{t-k}). \tag{B.14}$$

Then $\Phi(k) = \sum_{j=k}^{\infty} b_j b_{j-k}$. Thus the autocovariance of h_t coincides with that of x_t. A simple expression for $\Phi(k)$ can be obtained in $L^2[-\pi,\pi]$:

$$\begin{aligned}\Phi(k) &= E(h_0 \bar{h}_{0-k}) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h_0(\lambda)\overline{h_0(\lambda)\exp(-i\lambda k)}d\lambda \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h_0(\lambda)\overline{h_0(\lambda)}\exp(i\lambda k)d\lambda\end{aligned}$$

Hence

$$\Phi(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\exp(i\lambda k)d\lambda \tag{B.15}$$

where

$$\begin{aligned}f(\lambda) &= h_0(\lambda)\overline{h_0(\lambda)} \\ &= \left[\sum_{j=0}^{\infty} b_j \exp(i\lambda j)][\sum_{j=0}^{\infty} b_j \exp(-i\lambda j)\right]\end{aligned} \tag{B.16}$$

is the spectral density.

For a vector process $\mathbf{x}_t = B(L)\mathbf{v}_t = \sum_{j=0}^{\infty} B_j \mathbf{v}_{t-j}$ where $\mathbf{x}_t$ and $\mathbf{v}_t$ are $p \times 1$ vectors and B_j is a $p \times p$ matrix, we consider a matrix process $\mathbf{h}_t = \sum_{j=0}^{\infty} B_j \exp(i\lambda(t-j))$. Then define $\Phi(k) = E(\mathbf{x}_t\mathbf{x}'_{t-k})$ for $\mathbf{x}_t$ and $\Phi(k) = E(\mathbf{h}_t\bar{\mathbf{h}}'_{t-k})$ for $\mathbf{h}_t$. Then for both $\mathbf{x}_t$ and $\mathbf{h}_t$, $\Phi(k) = \sum_{j=k}^{\infty} B_j B'_{j-k}$, and

$$\Phi(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\exp(i\lambda k)d\lambda \tag{B.17}$$

where

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}\mathbf{h}_0(\lambda)\overline{\mathbf{h}_0(\lambda)}' \\ &= \sum_{j=0}^{\infty} B_j \exp(i\lambda(t-j)). \end{aligned} \tag{B.18}$$

B.4 Lag Operators

In this section, we will apply the results of the previous sections to polynomials and series of the lag operator. We will first consider stochastic processes of random variables. Then we will consider stochastic processes of random vectors.

Given $x_t = b(L)e_t = \sum_{j=0}^{\infty} b_j e_{t-j}$ with an orthonormal e_t in $L^2(Prob)$, we consider $h_t(\lambda) = \sum_{j=0}^{\infty} b_j \exp(-i\lambda(t-j)) = \sum_{j=0}^{\infty} b_j \exp(i\lambda j)\exp(-i\lambda t) = h_0(\lambda)\exp(-i\lambda t)$ in $L^2[-\pi,\pi]$ as in the previous section. As we will see, there is a one-to-one mapping that preserves the distance between the closed linear space generated by $\{u_t\}_{t=-\infty}^{\infty}$ and that by $\{e_t\}_{t=-\infty}^{\infty}$. Moreover, $L^n h_t(\lambda) = h_{t-n}(\lambda) = h_t \exp(i\lambda n)$. Hence applying the lag operator n times to the stochastic process h_t corresponds with multiplying a complex number h_t by $\exp(i\lambda)$ n times. For these reasons, we can study various properties of $b(L)e_t$ by studying the power series $b(z) = \sum_{j=0}^{\infty} b_j z^j$ of a complex variable z. For example, if $b(z)$ converges on the unit circle, then $b(\exp(i\lambda))$ converges for each λ in $[-\pi, \pi]$. This point-wise convergence in turn implies the convergence of $h_0(\lambda) = b_j \exp(i\lambda j)$ in $L^2[-\pi, \pi]$ by the Bounded Convergence Theorem.

Let $\{e_t\}_{t=-\infty}^{\infty}$ be an orthonormal sequence in $L^2(Prob)$, and let a sequence $\{u_t\}_{t=-\infty}^{\infty}$ in $L^2[-\pi, \pi]$ be defined by $u_t(\lambda) = \exp(-i\lambda t) = \cos(\lambda t) - i \sin(\lambda t)$ as in the previous section. Given $b(L)e_t = \sum_{j=0}^{\infty} b_j e_{t-j}$, consider a process $b(L)u_t = \sum_{j=0}^{\infty} b_j u_{t-j}$ in $L^2[-\pi, \pi]$. From Proposition 3.A.5, $b(L)e_t$ and $b(L)u_t$ converge if and only if $\{b_j\}$ is square summable. Hence $b(L)e_t$ converges if and only if $b(L)u_t$ converges.

From these results, we obtain the following proposition which gives a convenient sufficient condition for $b(L)e_t$ to be defined.

Proposition B.4.1 Let $\{e_t\}_{t=-\infty}^{\infty}$ be a white noise stochastic process with $E(e_t^2) = 1$. Suppose that $b(z) = \sum_{j=0}^{\infty} b_j z^j$ converges for $z = z_1$ such that $|z_1| > 1$. Then $\sum_{j=0}^{N} b_j e_{t-j}$ converges in mean square to a random variable with a finite second moment y_t as $N \to \infty$ and $y_t = b(L)e_t$ is a covariance stationary process.

Proof From Proposition B.1.5, $b(z) = \sum_{j=0}^{\infty} b_j z^j$ is converges for $|z| = 1$, and hence $b(\exp(i\lambda))$ converges for each λ in $[-\pi, \pi]$ in C. Let $s_N(\lambda) = \sum_{j=0}^{N} b_j \exp(i\lambda j)$, and let $h_0(\lambda) = b(\exp(i\lambda))$ be the limit of $s_N(\lambda)$ in C. For each λ, $\lim_{N\to\infty} |s_N(\lambda) - h_0(\lambda)| = 0$. Hence by Lebesgue's dominated convergence theorem,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |s_N(\lambda) - h_0(\lambda)| = \int_{-\pi}^{\pi} \lim_{N\to\infty} |s_N(\lambda) - h_0(\lambda)| = 0,$$

which implies that $\sum_{j=0}^{N} b_j \exp(-i\lambda j)$ converges in $L^2[-\pi, \pi]$ to $h_0(\lambda)$. Hence $\{b_j\}_{j=0}^{\infty}$ is square summable. To see that y_t is covariance stationary, note that $E(y_t) = b_0 E(e_t)$ does not depend on t. Since the inner product in L^2 is continuous, $E(y_t y_{t-\tau}) = \lim_{N\to\infty} E((\sum_{j=0}^{N} b_j e_{t-j})(\sum_{j=0}^{N} b_j e_{t-j-\tau}))$. Since e_t is covariance stationary, $E((\sum_{j=0}^{N} b_j e_{t-j})(\sum_{j=0}^{N} b_j e_{t-j-\tau}))E((\sum_{j=0}^{N} b_j e_{t-j})^2)$ does not depend on t. ∎

In this book, $b(L)e_t$ is taken to mean the limit of $\sum_{j=0}^{N} b_j e_{t-j}$ in $L^2(Prob)$ as $N \to \infty$.

Example B.4.1 Let $b(L) = 1 + aL + a^2L^2 + \cdots + a^nL^n + \cdots$. If $|a| < 1$, then $b(z)$ converges for $z = z_1$ where z_1 is a real number such that $1 < z_1 < a^{-1}$. Hence $b(L)e_t$ can be defined in $L^2(Prob)$. ∎

Proposition B.4.1 gives a sufficient condition for $b(L)e_t$ to be covariance stationary. The next proposition gives a sufficient condition for $b(L)e_t$ to be strictly stationary.

Proposition B.4.2 Let $\{e_t\}_{t=-\infty}^{\infty}$ be a strictly stationary white noise process with finite second moments. Suppose that $b(z) = \sum_{j=0}^{\infty} b_j z^j$ converges for $z = z_1$ such that $|z_1| > 1$. Then $y_t = b(L)e_t$ is a strictly stationary process.

Proof Let $s_{Nt} = \sum_{j=0}^{N} b_j e_{t-j}$. Then s_{Nt} converges in mean square to y_t as $N \to \infty$. Therefore, s_{Nt} converges in probability to y_t, and hence it converges in distribution to y_t. Let $F_{Nt}(\zeta)$ be the distribution function of s_{Nt}, and $F_t(\zeta)$ be the distribution function of y_t. Then $F_{t+\tau}(\zeta) = \lim_{N\to\infty} F_{Nt}(\zeta) = F_t(\zeta)$ except for the discontinuity points of $F_{t+\tau}(\zeta)$ and $F_t(\zeta)$ because e_t is strictly stationary. There are only countably many discontinuity points, and the distribution function is right continuous. Therefore, $F_{t-\tau}(\zeta) = F_t(\zeta)$ for all ζ. Similar arguments can be made to show that the joint distribution function of $(y_t, y_{t+1}, \cdots, y_{t+k})$ does not depend on t. ∎

References

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